分析

NCU PHD PROGRAM ENTRANCE EXAM: ANALYSIS

(May 14, 2010)

Stage Setting: In the following problems, whenever not specified, the functions are assumed be real-valued.

(1) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$ with $a_n, b_n \geq 0$ for all n. Prove that

$$\liminf a_n \cdot \liminf b_n \leq \liminf(a_n b_n). \quad (10\%)$$

(2) Let $f : [0,1] \to \mathbb{R}$ be defined by setting $f(0) = 0$ and $f(x) = x^3 \cos(1/x^2)$ if $x \in (0,1]$. Determine whether f is absolutely continuous on $[0,1]$. Prove your answer. (10%)

(3) Let $f, f_1, f_2, \cdots$ be measurable functions on the measure space $(X, \mathcal{B}, \mu)$, and, $f_n \geq f_{n+1}$ for $n = 1, 2, \cdots$.

(a) If $f_n \to f$ in measure, prove that $f_n \to f$ almost everywhere. (10%)

(b) If $f_n \to f$ almost everywhere, prove or disprove that $f_n \to f$ in measure. (5%)

(4) For each n let the function $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx^{n-1}}{1+x}$ for all $x \in [0,1]$. Then show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2}$. (10%)

(5) Consider a measurable space $(X, \mathfrak{B})$. Suppose $\{\mu_n\}$ is a sequence of measures on $(X, \mathfrak{B})$ with μ_n increasing to a measure μ, that is, $\mu_n(A)$ is increasing to $\mu(A)$ for every $A \in \mathfrak{B}$. Then for every nonnegative measurable function f, prove that $\int_X f d\mu_n$ is increasing to $\int_X f d\mu$. (15%)

(6) Let $1 < p < \infty$ and $f \in L^1(\mathbb{R})$, $g \in L^p(\mathbb{R})$. For each $x \in \mathbb{R}$, let $h_x(y) = f(x-y)g(y)$, $y \in \mathbb{R}$. Prove that h_x is Lebesgue integrable on $\mathbb{R}$ and the function $H(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$ is in $L^p(\mathbb{R})$ with $\|H\|_p \leq \|f\|_1 \|g\|_p$. (15%)

(7) Let $C = C([0,1], \mathbb{R})$ be the normed space of all continuous real-valued functions on $[0,1]$ with the norm $\|f\|_1 = \int_0^1 |f(x)|dx$. Let

$U = \{f \in C : \|f\|_\infty \leq 1\}$;

$V =$ the closure of the set $\{p \in C : p \text{ is a polynomial}\}$ in C;

$W = \{f \in C : 1 < \int_0^1 f(x)dx < 2\}$;

$X = \{f_0, f_1, f_2, f_3, \cdots\}$, where $f_n \in C$ for all $n \geq 1$ and $f_n \to f_0$ uniformly on $[0,1]$;

$Y =$ the closure of A in C, where

$A = \{f \in C : f \text{ is differentiable on } (0,1), f(0) = 1 \text{ and } |f'| \leq 2\}$.

(a) Which sets are compact in C? If the set is compact, give your proof. (10%)

(b) Which sets are complete in C? (Don't need to prove it.) (5%)

(c) Prove that the integral equation

$$f(x) = \int_0^x e^{-xy} f(y)dy, \quad x \in [0,1],$$

has a unique solution $f \in U$ (10%)