一　　　　代數

國立中央大學數學研究所博士班入學考試

科目：代數

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1. (25 %) Let k be a field and let $k[x]$ be the ring of polynomials in x with coefficients in k. Let M be a $k[x]$-module.
   (a) Let $\widetilde{M}$ be the sum of all $k[t]$-submodule V such that $\dim_k V < \infty$. Prove that $\widetilde{M} = M_{\text{tor}}$ where M_{tor} denotes the torsion $k[t]$-submodule of M.
   (b) Suppose that M_{tor} is a direct sum of cyclic modules $N_1, \ldots, N_4$ whose annhilators are the ideals $I_j, j = 1,2,3,4$ generated by the polynomials $p_1(t)^{l_1}, p_2(t)^{m_1}$, $p_1(t)^{l_2}p_2(t)^{m_2}$ and $p_3(t)^n p_2(t)^{m_3}$ with $l_1 \leq l_2$ and $m_1 \geq m_2 \geq m_3$. Here $p_1(t), p_2(t), p_3(t)$ are distinct irreducible polynomials. Compute the invariants of M_{tor}. (The invariants of a finitely generated torsion module N is a decreasing sequence of ideals $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_r$ of $k[x]$ such that $N \simeq \bigoplus_{i=1}^{r} k[x]/\mathfrak{q}_i$.)
   (c) Continuing the assumption in (b). Let $T : M_{\text{tor}} \to M_{\text{tor}}$ be given by $T(m) = x * m$ for all $m \in M_{\text{tor}}$. Then T is a linear transformation on the finite dimensional k-vector space M_{tor}. Determine the characteristic polynomial and the minimal polynomial of T.

2. (20 %) Let p be an odd prime number. Let L be the splitting field of $g(x) = (x^p - q_1)\cdots(x^p - q_r)$ over $\mathbb{Q}$ where $q_1, \ldots, q_r$ are distinct prime numbers. Let G be the Galois group of L over $\mathbb{Q}$. Let Q be a p-Sylow subgroup of G.
   (a) Prove or disprove the following statement: Q is an abelian, normal subgroup of G and the quotient G/Q is cyclic.
   (b) Compute the order of Q.

3. (20 %) Let p be a prime number. Let S_p be the group of permutations of a set of p elements.
   (a) Let $N_P = \{\sigma \in S_p \mid \sigma P \sigma^{-1} = P\}$ be the normalizer of P in S_p. Show that N_P is isomorphic to the semi-direct product of P and $\mathrm{Aut}(P)$ where $\mathrm{Aut}(P)$ denotes the automorphism group of P.
   (b) Determine the number of p-Sylow subgroup in S_p.

4. (20 %)

(a) Let Γ be a free abelian group of rank $n \geq 1$. Let Γ' be a subgroup of Γ which is of rank n also. Let $\{v_1, \dots, v_n\}$ be a basis of Γ, and let $\{w_1, \dots, w_n\}$ be a basis of Γ'. Write

$$w_i = \sum a_{ij} v_j, \quad a_{ij} \in \mathbb{Z}.$$

Show that the index $[\Gamma : \Gamma']$ is finite and is equal to the absolute value of the determinant of the matrix $A = (a_{ij})$.

(b) Let R be a principal ideal domain. Let E be a free module of rank n over R, and let $E^\vee = \mathrm{Hom}_R(E, R)$ be its dual module. Show that $E^\vee$ is a free module over R of rank equal to n.

5. (15 %) Let K be a field and let $|\cdot| : K \to \mathbf{R}$ be an absolute value satisfying the strong triangle inequality. That is, it satisfies (i) $|x| \geq 0$ for all $x \in K$ and $|x| = 0$ if and only if $x = 0$; (ii) $|xy| = |x||y|$ for all $x, y \in K$ and (iii) $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$. Let $K[x]$ be the polynomial ring over K. For any polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in K[x]$, define the norm of $f(x)$ by the formula:

$$\|f\| := \max\{|a_i| \mid i = 0, \dots, n\}.$$

Prove or disprove the following statement:

For any two polynomials $f(x), g(x) \in K[x]$, we always have $\|fg\| = \|f\| \, \|g\|$ where $(fg)(x) = f(x)g(x)$ is the product of the two polynomials f and g.