1. In this problem we consider integral inequality.

(a) (10%) Suppose that $x(t)$ is a continuous function satisfying

$$x(t) \le M_0 + \int_0^t a(s)x^2(s)ds \qquad \forall\, t \ge 0\,,$$

for some $a \in L^2(0,\infty)$ and constant M_0. Show that there is a $T > 0$, independent of x, such that $x(t) \le 2M_0$ for all $t \in [0,T]$.

(b) (10%) Let $\mathcal{P}$ be a polynomial of one variable, and $x(t)$ be a non-negative, continuous function satisfying the following inequality

$$x(t) \le M_0 + b(t)\mathcal{P}(x(t)) \qquad \forall\, t \ge 0\,,$$

where M_0 is again a constant, and $b(t)$ is a continuous function with $b(0) = 0$. Show that

$$x(t) \le 2M_0 \qquad \forall\, t \in [0,T]$$

for all small enough $T > 0$.

2. Let us consider the BBM equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad \forall\, x \in \mathbb{R}\,, t \in (0,T]\,, \tag{1a}$$

$$u(x,0) = g(x) \qquad \forall\, x \in \mathbb{R}\,. \tag{1b}$$

(a) (10%) Given that

$$\int_{-\infty}^{\infty} e^{-|x|-ikx}dx = \frac{4}{1+k^2}\,,$$

use the Fourier transform to show that a bounded solution to (1) satisfies

$$u(x,t) = g(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\Big[u(y,s) + \frac{1}{2}u^2(y,s)\Big]dyds\,, \tag{2}$$

where K is defined by

$$K(x) = \text{sign}(x)e^{-|x|}\,.$$

(b) (10%) Write (2) as $u = F(u)$, that is, treat the right-hand side of (2) as a functional of u. Show that for $T > 0$ small enough, F has a fixed-point in the space of bounded continuous functions. (Hint: similar to the proof of the fundamental theorem of ODE, you can try to show that the map F is a contraction mapping if T is small enough, and then apply the contraction mapping theorem.)

3. Consider the system

$$x' = -x - xy^2\,,$$
$$y' = -x^2y\,.$$

(a) (10%) Linearize the system about the rest point $(0,0)$, and determine the stability of this rest point for the linearized system.

(b) (10%) Prove that the rest point $(0,0)$ is *stable* for this nonlinear system.

4. (10%) The Lorentz Equations are given by

$$x' = -\sigma x + \sigma y\,,$$
$$y' = -xz + rx - y\,,$$
$$z' = xy - bz\,,$$

here σ, r and b are positive constants. Prove that when $0 < r < 1$, there exists only one rest point $(0,0,0)$, and all solution trajectories tend to this stable rest point as $t \to \infty$. Prove this by stating the Lyapunov stability theorem, and then show that $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$ is a Lyapunov function when $0 < r < 1$.

5. In this problem we consider the system

$$x' = -x\,,$$
$$y' = y + x^2\,.$$

(a) (10%) Find the stable and unstable manifold $W^s(0)$ and $W^u(0)$ to the system.

(b) (10%) State the stable manifold theorem and check the validity of the theorem for $W^s(0)$ and $W^u(0)$.

6. (10%) Suppose that Ω is a smooth, bounded, connected open set in $\mathbb{R}^2$, and $u(x,t)$ is the solution to the Euler equations

$$u_t + u \cdot \nabla u = -\nabla p + f \qquad \text{in} \quad \Omega \times (0,T]\,, \tag{3a}$$
$$\operatorname{div} u = 0 \qquad \text{in} \quad \Omega \times (0,T]\,, \tag{3b}$$
$$u \cdot n = 0 \qquad \text{on} \quad \partial\Omega \times (0,T]\,, \tag{3c}$$
$$u(x,0) = u_0(x) \qquad \forall\, x \in \Omega\,, \tag{3d}$$

where n is the outward-point unit normal of $\partial\Omega$. It is well-known that for smooth initial data u_0 and forcing f, there exists a unique smooth solution u for all $T > 0$. The streamlines are defined as the trajectories of particles following the fluid velocity. Assume that there is an time-independent, non-vanishing, smooth solution u to (3) (this is true only if f is time-independent). Prove or disprove that all the streamlines are closed, that is, all the streamlines are periodic orbits.