微分

NCU PH.D. ENTRANCE EXAM - DIFFERENTIAL GEOMETRY May 14, 2010

Show your work in details. In each problem, you may assume the former part(s) and work on the latter part(s) directly.

1. (a) Consider the torus of revolution generated by rotating the circle

$$(x-a)^2 + z^2 = r^2, \; y = 0$$

about the z-axis ($a > r > 0$). Show that the circle generated by the point $(a+r, 0) \in \mathbb{R}^2_{x,z}$ is a geodesic on the torus of revolution.

(b) State Gauss-Bonnet Theorem for an orientable compact surface.

(c) Compute the Euler-Poincaré characteristic $\chi(T)$ of a torus T via a triangulation of T.

(d) Is it possible to embed T in $\mathbb{R}^3$ so that its Gaussian curvature $K \equiv 0$ in the induced metric? Justify your answer.

(e) Give an example of a flat torus and verify that its curvatrure $K \equiv 0$.

2. Let $\alpha := dz + xdy - ydx \in \Omega^1(\mathbb{R}^3)$. Consider the distribution $E \subset TM$ defined by

$$E_p := \{v_p \in T_p\mathbb{R}^3 \mid \alpha_p(v_p) = 0\}, \quad p \in \mathbb{R}^3.$$

Determine whether or not E is integrable. Prove your answer.

3. Show that the form

$$\omega = \Big(\frac{-y}{x^2+y^2}\Big)dx + \Big(\frac{x}{x^2+y^2}\Big)dy$$

defined on the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ is closed but not exact.

4. Let M_1 and M_2 be Riemannian manifolds, and consider the product $M := M_1 \times M_2$, with the product metric. For $i = 1, 2$ let $\pi_i : M_1 \times M_2 \to M_i$ be the standard projection. Let ∇^i be the Riemannian connection of M_i for $i = 1, 2$. Let ∇ be the Riemannian connection of M.

(a) Show that if vector fields X, Y on M are of the form

$$X = \pi_1^* X_1 + \pi_2^* X_2, \quad Y = \pi_1^* Y_1 + \pi_2^* Y_2,$$

with X_i, Y_i being vector fields on M_i, then

$$\nabla_X Y = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2.$$

(b) Prove that the secitonal curvature of the Riemannian manifold $S^2 \times S^2$ with the product metric, where S^2 is the unit sphere in $\mathbb{R}^3$, is non-negative.

5. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$. Consider the antipodal map

$$A : S^n \to S^n, \quad A(x) := -x.$$

(a) Show that S^n is orientable.

(b) Show that A is orientation preserving if n is odd, and orientation reversing if n is even.

(c) Determine the degree $deg(A)$ of the map A for each value of n.