圖論

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In the following all the graphs considered are finite simple graphs.

1. (10%) Let $k(G)$ and $k'(G)$ denote the connectivity and edge connectivity respectively of a graph G. Suppose that G is a connected graph. Show that $k(G) \le k'(G)$.

2. Let G be a graph of order $n \ge 3$.
   (10%) (1) Suppose that $\deg x + \deg y \ge n$ for every pair of nonadjacent vertices x, y. Show that G contains a Hamiltonian cycle.

   (5%) (2) Suppose that $\deg x + \deg y \ge n-1$ for every pair of nonadjacent vertices x, y. Show that G contains a Hamiltonian path.

   (5%) (3) Suppose that G contains at least $\binom{n-1}{2} + 2$ edges. Show that G contains a Hamiltonian cycle.

3. (15%) (1) Show that for any positive integers k, there exists a graph with chromatic number k which contains no triangle.

   (5%) (2) Show that for positive integers $k \ge l \ge 2$, there exists a connected graph with chromatic number k and clique number l. (Clique number of a graph is the maximum size of a clique (complete graph) contained in a graph.)

4. (15%) $A_1, A_2, \cdots\cdots, A_n$ are distinct subsets of $\{1,2,\cdots\cdots,n\}$. Show that there exists $k \in \{1,2,\cdots\cdots,n\}$ such that $A_1 - \{k\}, A_2 - \{k\}, \cdots\cdots, A_n - \{k\}$ are all distinct.

5. (15%) (1) Let G be a bipartite graph with bipartition (X,Y) such that $|N(A)| \ge |A|$ for every $A \subset X$. Show that G has a matching M with $|M| = |X|$
   (5%) (2) Let G be a regular bipartite graph. Show that G has a perfect matching.

6. (15%) Show that every digraph D contains an independent set S such that each vertex of D is reachable from a vertex in S by a directed path of length at most two.