所別：數學系碩士班 乙組(一般生) 科目：數值分析

Instructions: Do all 5 problems. Show your work.

1. **(Iterative methods)**

   Let A be an $n \times n$ tridiagonal matrix in the form

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

   (a) (8 pts) Show that the eigenvalues of A are given by

$$\lambda_j = 2 - 2\cos(j\theta), j = 1, ..., n,$$

   and the eigenvector associated with each λ_j is given by

$$V_j = [\sin(j\theta), \sin(2j\theta), ..., \sin(nj\theta)]^T,$$

   where $\theta = \frac{\pi}{n+1}$.

   (b) Let us consider the problem of solving $Ax = b$ using iterative methods. One of most popular methods is called the Jacobi method, which can be written as

$$x^{(k+1)} = G_J x^{(k)} + D^{-1}b,$$

   where

$$G_J = D^{-1}(C_L + C_U).$$

   Note that D is the diagonal component of A, C_L is lower triangular part of A, and C_U is upper triangular part of A. Now consider *the weighted Jacobi method* given by

$$x^{(k+1)} = G_{WJ} x^{(k)} + \omega D^{-1}b,$$

   where $G_{WJ} = (1-w)I + \omega G_J$. Show that (i) G_{WJ} can be expressed as $G_{WJ} = I - \frac{w}{2}A$ (6 pts) and that (ii) the weighted Jacobi method will converge with any initial guess provided that $0 < w \leq 1$ (6 pts) .

注意：背面有試題

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2. (**Numerical Differentiation**) Suppose an exact quantity L is approximated by the function $N_1(h)$ according to the following relationship

$$L = N_1(h) + K_1 h^{1/2} + K_2 h^{2/2} + K_3 h^{3/2} + \cdots.$$

(a) (4 pts) What is the rate of convergence of $N_1(h)$?

(b) (12 pts) Use a combination of $N_1(h)$ and $N_1(h/2)$ to derive an improved approximation $N_2(h)$.

(c) (4 pts) What is the rate of convergence of $N_2(h)$?

3. (**Numerical Integration**)

(a) (10 pts) Find A_0, A_1, and A_2 such that

$$\int_{-1}^{1} f(x)\,dx \approx A_0 f(-1/2) + A_1 f(0) + A_2 f(1/2)$$

is **EXACT** for all polynomials of degree ≤ 2.

(b) (10 pts) Derive a corresponding rule for an arbitrary interval $[a, b]$.

4. (**Interpolation**) One naive procedure to construct a polynomial interpolation passing the data (x_i, y_i), $0 \leq i \leq n$ is as follows: First consider

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Then the interpolation conditions, $p(x_i) = y_i$ for $0 \leq i \leq n$, lead to a system of $n+1$ linear equations for determining $a_0, a_1, \ldots a_n$:

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

The coefficient matrix is called the **Vandermonde matrix**. (i) Explain why this approach is not recommended in practice (10 pts) and (ii) suggest a better algorithm for constructing the interpolation (please state why your algorithm is better than the algorithm described above) (10 pts).

5. (**Eigenvalue problems**) Consider

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 15 & 3 \\ 0 & 0 & 0 & 3 & 25 \end{pmatrix}$$

(a) (10 pts) Estimate the location of the second largest eigenvalue of A.

(b) (10 pts) Write down completely an algorithm based on the **power method** to compute this eigenvalue.