ALGEBRA

February 2004

1. (20%) Let G be a non-abelian group of order p^3 with p a prime.

(a) Show that its center $Z(G) \simeq \mathbf{Z}/p\mathbf{Z}$ and $G/Z(G) \simeq \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$,

(b) Let H be a subgroup of order p^2. Show that $H \supset Z(G)$ and H is normal.

(c) Show that if $a^p = 1$ for all $a \in G$, then there exists a normal subgroup $H \simeq \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$.

(d) Show that there are at most two non-abelian groups of order p^3.

2. (20%) Let $R = \mathbf{Z}[\zeta]$, where $\zeta = (-1+\sqrt{-3})/2$ is a complex cube root of 1. Let p be an integer prime.

(a) Show that R is a UFD.

(b) Show that (p) is a prime ideal of R if and only if $p \equiv -1 \pmod 3$.

(c) Show that p factors in R if and only if it can be written in the form $p = a^2 - ab + b^2$ for some integers a, b.

(d) Find the primes of absolute value ≤ 7 in R.

3. (10%) If R is Noetherian, then $R[x]$ is also Noetherian.

4. (15%)

(a) Let G be an abelian group and $G = \langle x, y, z, u : 6x + 9y = 12x = 8z + 12u = 0\rangle$. Please write G as a direct sum of cyclic groups.

(b) Show that any submodule of a free module of finite rank over a PID is free. In particular, every finitely generated projective module over a PID is free.

5. (20%)

(a) For each positive integer n, there exists an extension L of a field K such that $\mathrm{Gal}(L/K) \simeq S_n$.

(b) Is every finite group isomorphic to some Galois group $\mathrm{Gal}(F/K)$ for some extension F of some field K? Justify your answer.

6. (15%)

(a) Show that if L is an algebraic extension over a field F and D is a domain such that $F \subset D \subset L$, then D is a field.

(b) Show that an algebraically closed field is infinite.

(c) Let $f(x_1, x_2, \ldots, x_n) \neq 0 \in k[x_1, x_2, \ldots, x_n]$ with k an algebraically closed field. Show that $f = 0$ has a solution in k^n. What if k is not algebraically closed?