ALGEBRA (代數)

September 2005

1. (20%) Show that there are five isomorphism classes of groups of order 12. They are represented by

(a) the cyclic group $\mathbf{Z}_{12}$;
(b) the product of cyclic groups $\mathbf{Z}_2 \times \mathbf{Z}_6$;
(c) the alternating group A_4;
(d) the dihedral group D_6;
(e) the group generated by x, y, with relations $x^4 = 1$, $y^3 = 1$, $xy = y^2x$.

2. (20%) Let K be an arbitrary field and let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m \in K[x]$ with $a_0b_0 \neq 0$.

(a) Show that $f(x)$ and $g(x)$ have a common nonconstant factor if and only if their resultant $R(f,g)$ is zero.
(b) Assume $f(x) = a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$. Show that

$$R(f, f') = a_0^{2n-1}\prod_{i\neq j}(\alpha_i - \alpha_j),$$

where f' is the derivative of f. Moreover, $f(x)$ has a multiple root if and only if $R(f,f') = 0$.

3. (20%)

(a) Show that any $\mathbf{Z}$-module can be imbedded in an injective $\mathbf{Z}$-module. (Bonus: You get extra 10 points by showing that any module can be imbedded in an injective module.)
(b) Assuming (a), show that any module has a unique injective resolution up to homotopy equivalence.

4. (20%) Let K be a finite separable extension of a field k. Show that there exists an element $\alpha \in K$ such that $K = k(\alpha)$. (Hint: Consider that k is finite or k is infinite separately.)

5. (10%) Let p be a prime integer and K be the splitting field of $x^p - 1$ over $\mathbf{Q}$. Determine the Galois group $\mathrm{Gal}(K/\mathbf{Q})$. (Explain your answer in detail.)

6. (10%) If R is Noetherian, then $R[x]$ is also Noetherian.