Algebra

Qualify Exam., Spring 2005

(20%) 1. Let G be a finite group of order $2pq$, where p, q are odd primes with $p \leq q$. Prove that G is solvable.

(15%) 2. Let $R = \mathbb{Z}[\sqrt{-2}]$, and let $m, n \in \mathbb{Z}$.

(a) Prove that R is a unique factorization domain.

(b) Prove or disprove: if $m^2 + 2n^2$ is a prime in $\mathbb{Z}$, then $m + n\sqrt{-2}$ is a prime element in R.

(c) Prove or disprove: if $m + n\sqrt{-2}$ is a prime element in R, then $m^2 + 2n^2$ is a prime in $\mathbb{Z}$.

(15%) 3. (a) Prove that the following two properties are equivalent.

(1) Every algebraic extension of K is separable.

(2) Either $char(K) = 0$, or $char(K) = p$ and every element of K has a p-th root in K.

(b) Prove that every algebraic extension of a finite field is separable.

(15%) 4. (a) Denote K to be the splitting field of the polynomial $f(x) = x^3 + 9x + 9$ over $\mathbb{Q}$. Find the Galois group of K over $\mathbb{Q}$. And determine all the intermediate fields between K and $\mathbb{Q}$.

(b) Denote F to be the splitting field of the polynomial $f(x) = x^3 - 2005x + 217$ over $\mathbb{Z}/3\mathbb{Z}$. Find the Galois group of F over $\mathbb{Z}/3\mathbb{Z}$. And determine all the intermediate fields between F and $\mathbb{Z}/3\mathbb{Z}$.

(20%) 5. Let R be a commutative ring and let M be an R-module. M is said to be a flat R-module if the induced sequence $0 \longrightarrow M' \otimes R \longrightarrow M \otimes R$ is exact provided the sequence $0 \longrightarrow M' \longrightarrow M$ is exact. For a prime ideal $\wp$ of A, denote $M_{\wp} = S^{-1}M$, where $S = A \setminus \wp$. Prove that M is flat if and only if the localization $M_{\wp}$ is flat over $R_{\wp}$ for each prime ideal $\wp$ of R.

(15%) 6. Let V be a finite dimensional vector space over a field F, and $T : V \to V$ a linear transformation. Suppose that T^i has trace 0 for all $i \geq 1$.

(a) Suppose that F has characteristic 0. Prove or disprove that $T^m = 0$ for some positive integer m.

(b) Suppose that F has characteristic $p > 0$. Prove or disprove that $T^m = 0$ for some positive integer m.