2006年9月

國立中央大學數學系博士班資格考試試題

科目：代數

1. Describe the Galois groups of the following polynomials:

   (a) $x^3 - 3x + 1$.

   (b) $x^5 - 2$.

(10 pts)

2. Let $\mathbb{F}_{p^n}$ be a finite field of p^n elements, p is a prime number.

   (a) Explain why $\mathbb{F}_{p^n}/\mathbb{F}_p$ is a finite Galois extension.

   (b) Let $\sigma_p : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ be defined by $\sigma_p(x) = x^p$. Show that

$$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^n}) =< \sigma_p >$$

   (the cyclic group generated by σ_p).

(10 pts)

3. (a) Give the definition and an example of Noetherian ring.

   (b) State the Hilbert Basis Theorem.

   (c) State the fundamental structure theorem for finitely generated modules over a P.I.D..

(15 pts)

4. (a) Let $F \subset E$ be an algebraic fields extension. Show that any subring D of E containing F is a subfield of E.

   (b) Let $\mathbb{C}$ be the field of complex numbers. Show that

$$\bar{\mathbb{Q}} = \{\alpha \in \mathbb{C} | \alpha \text{ is algebraic over } \mathbb{Q}\}$$

   is a subfield of $\mathbb{C}$.

(10 pts)

5. (1) Is 2 an irreducible element in $\mathbb{Z}[\sqrt{-3}]$? Prove your answer.

   (2) Is 2 a prime element in $\mathbb{Z}[\sqrt{-3}]$? Prove your answer.

(15 pts)