ALGEBRA

Qualify Exam., August 2007

1. (25 %)

(a) Show that no group of order 224 is simple.

(b) Classify groups of order $4p$, where p is a prime greater than 3 and $p \equiv 3 \pmod 4$. Are they always solvable? (Justify your answer)

2. (25 %)

(a) Let I be the set of all non-units of $\mathbf{Z}[i]$, where $\mathbf{Z}$ is the ring of integers and $i = \sqrt{-1}$. Is I an ideal of $\mathbf{Z}[i]$? (Justify your answer)

(b) Show that for any non-trivial ideal J of $\mathbf{Z}[i]$, the quotient ring $\mathbf{Z}[i]/J$ is a finite ring.

(c) Determine all prime elements in $\mathbf{Z}[i]$. (Justify your answer)

3. (25 %)

(a) Let R be an integral domain with quotient field Q and N be any R-module. Show that $Q \otimes_R N = 0$ if and only if N is a torsion module.

(b) Show that tensor products do not commute with direct products in general.
(Hint : Consider the direct product of $M_i = \mathbf{Z}/p^i\mathbf{Z}$ with p a prime.)

(c) Show that a finitely generated module over a P.I.D. is projective if and only if it is free.

(d) Show that if R is a P.I.D but not a field, then no finitely generated R-module is injective.

4. (25 %)

(a) Is the equation

$$f(x) = x^5 - 10x^4 + 2x^3 - 24x^2 + 2 = 0$$

solvable by radical over $\mathbf{Q}$? (Justify your answer)
Here $\mathbf{Q}$ is the field of rational numbers.

(b) Let n be a positive integer and ξ be a primitive nth root of unity over $\mathbf{Q}$. Determine

$$[\mathbf{Q}(\xi + \frac{1}{\xi}) : \mathbf{Q}].$$

(Justify your answer)

(c) Let $A = \{\sqrt{p} \,|\, p \text{ is a prime}\}$ and $L = \mathbf{Q}(A)$. Show that $\mathrm{Gal}(L/\mathbf{Q})$ has uncountably many subgroups of index 2 and there are only countably many quadratic field extensions of $\mathbf{Q}$ in L.