國立中央大學數學系
博士班資格考試
〈代數〉試題
2007 年 2 月

(1) (15%) Consider a finite group G with an abelian normal subgroup $H \triangleleft G$ such that the quotient group $Q := G/H$ is also abelian. And consider the condition that for every $g \in G$, the automorphism $\Phi_g : H \longrightarrow H, h \mapsto ghg^{-1}$, is the identity map.

(a) Find a non-abelian group G that satisfies the above conditions.

(b) Show that if Q is cyclic then G must be abelian.

(2) (15%) Suppose K is a field and m is a positive integer relatively prime to the characteristic of K. Let $K(\zeta_m)/K$ denote the splitting field of $X^m - 1$.

(a) For the case where m is a prime number, show that the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^*$, the units group of the ring $\mathbb{Z}/m\mathbb{Z}$.

(b) Let $\mathbb{F}_q$ be a finite field with q elements. Then $\mathrm{Gal}(\mathbb{F}_q(\zeta_m)/\mathbb{F}_q) \simeq \mathbb{Z}/n\mathbb{Z}$, the cyclic group of order n, where n is the smallest natural number such that $q^n \equiv 1 \pmod{m}$.

(3) (10%) Suppose that L_1/K and L_2/K are abelian extensions (Galois extensions with abelian Galois groups). Is it always true that L_1L_2/K is also abelian? Prove or disprove it.

(4) (10%) Prove that if K is a field, then $K[X,Y]$, the polynomial ring in two variables, is a Noetherian ring.

(5) (30%) Let $A = \mathbb{C}[X]$ and $K = \mathbb{C}(X)$.

(a) Suppose $p(X) \in \mathbb{Q}[X]$, $\deg(p(X)) = d$, is irreducible over $\mathbb{Q}$. Show that the ring $A/(p(X))$ is isomorphic to the direct sum of d-copies of $\mathbb{C}$.

(b) An extension L/K with $|L : K| = 3$ is a Galois extension if and only if $L = K(\alpha^{\frac{1}{3}})$ for some $\alpha \in K^*$.

(c) Let $\mathcal{O}$ be the integral closure of A in a quadratic extension $K(\beta^{\frac{1}{2}})/K$, where $\beta \in A$ is square free in the sense that if $p(X)$ is an irreducible