Algebra 2009-Fall

1. Let G be any group of order 203. Prove or disprove that G is a cyclic group. (15%)

2. Let R be an integral domain, let F be its field of fractions and let $p(x)$ be a monic polynomial in $R[x]$. Prove or disprove that $p(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. (10%)

3. Let R_i be a principal ideal domain with identity $1 \neq 0$ for $i \in \boldsymbol{N}$, where $\boldsymbol{N}$ is the set of all positive integers. Prove or disprove that the direct product $\prod_{i \in N} R_i$ is not a principal ideal ring. (15%)

Definition. Let K/F be an extension of fields and $Aut_F K$ be the group of automorphisms of K which fix F. Then K is said to be a Galois extension of F if F is the fixed field of $Aut_F K$.

4. Let K/F be any finite field extension. Prove or disprove that $|Aut_F K| \leqq [K:F]$. (15%)

5. (a) Give the definition of a perfect field. (5%)

   (b) Give (with proof) a field which is not a perfect field. (10%)

6. Let R be a ring with identity $1 \neq 0$, N be a left R-module and M be a right R-module with a right R-submodule M_0. Prove or disprove that the tensor product $M_0 \otimes_R N$ is a subgroup of $M \otimes_R N$. (10%)

7. Let K be a Galois extension of field F. Prove or disprove that K is an algebraic extension of F. (10%)

8. Let K/F be an extension of fields with $[K:F] = 2$. Prove or disprove that K is a Galois extension of F. (10%)