代數 資格考 Feb. 2009

(20%) 1. For a group G and a prime number p, define the p-part G_p of G as follows:

$$G_p = \{x \in G : \text{ the order of } x \text{ is a power of } p\}$$

A group G is said to be a p-group $G = G_p$.

(a) Prove or disprove that, for a finite group G, G is a p-group if and only if the order of G is a power of p.

(b) Prove or disprove that G is finite if and only if G_p is finite for all primes p.

(c) Prove that every finite p-group G has a nontrivial center.

(d) Prove that every finite p-group G is solvable.

(20%) 2. Let R be an integral domain.

(a) Find the group of units in $R[x]$, the ring of polynomials over R.

(b) Find the group of units in $R[[x]]$, the ring of formal power series over R.

(c) Suppose that R is a field. Prove or disprove that $R[x]$ is a unique factorization domain.

(d) Suppose that R is a field. Prove or disprove that $R[[x]]$ is a unique factorization domain.

(15%) 3. Let K be a field and $f(x) = x^3 - 3x + 1 \in K[x]$. K

(a) Prove that $f(x)$ is separable over K if and only if char($\cancel{K}$) $= 3$.

(b) Prove that $f(x)$ is either irreducible or splits into linear factors in K.

(c) Let L be a splitting field of $f(x)$ over K. Prove or disprove that $[L : K]$ divides 3.

(15%) 4. Let F be a finite field with $\text{char}(F) = p > 0$ and let $\bar{F}$ be algebraic over F and algebraically closed. For a positive integer n, denote F_n to be the spiltting field of $x^n - 1$ in $\bar{F}$.

(a) Show that $\bar{F} = \cup_{n \geq 1} F_n$ and $[\bar{F} : F] = \infty$.

(b) Show that for any positive integer n with $p \mid n$, $F_n = F_{n/p}$.

(c) Prove or disprove that, for any positive integers m and n, $F_m \cap F_n = F$ if and only if $\gcd(m, n)$ is a power of p.

(30%) 5. Let R be a principal ideal domain and M a finitely generated R-module.

(a) For $S = R\backslash\{0\}$, prove that $M \underset{R}{\otimes} S^{-1}R$ is a vector space over $S^{-1}R$ and is isomorphic to $S^{-1}M$. Show that $\text{rank}_R(M) = \dim_{S^{-1}R}\left(S^{-1}M\right)$.

(b) For a prime ideal $\wp$ of R, denote $M_\wp = S^{-1}M$, where $S = A \backslash \wp$. Prove or disprove that M is projective R-module if and only if $M_\wp$ is projective over $R_\wp$ for each prime ideal $\wp$ of R.