國立中央大學數學系
博士班資格考試〈代數〉試題

ALGEBRA

SEPTEMBER, 2002

1. a. (6 pts.) Show that every group of order 15, 51 or 85 is cyclic.

   b. Let G be a group of order 255.

      (1) (3 pts.) Show that there is only one subgroup of order 17 in G.

      (2) (3 pts.) Show that there is a subgroup having either order 3 or order 5 and normal in G.

      (3) (5 pts.) Let C be the commutator subgroup of G. Show that $C = \{e\}$, where e is the identity element of G.

      (4) (3 pts) Show that G is cyclic.

2. a. (8 pts.) Let G be an abelian group and $G = \langle x, y, z, u : 6x + 9y = 12x = 8z + 12u = 0\rangle$. Please write G as a direct sum of cyclic groups.

   b. (12 pts.) Show that every finitely generated projective module over a PID is free.

3. a. (10 pts.) (Nakayama's lemma) Let A be a commutative ring with identity. Let M be a finitely generated A-module and I an ideal of A contained in the Jacobson radical of A. Show that $IM = M$ implies $M = 0$.

   b. (10 pts.) Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

4. a. (10 pts.) Determine all the subgroups of the Galois group and all of the intermediate fields of the splitting field (over $\mathbb{Q}$) of the polynomial $X^4 - 12X^2 + 25$.

   b. (10 pts.) What is the Galois group over $\mathbb{Q}$ of the polynomial $X^5 - 10X + 5$? Is it solvable?