國立中央大學數學系
博士班資格考試
〈代數〉試題
2002 年 2 月

In the following, we let R be a commutative ring with identity.

(1) Let M be a R-module.

(i)(5 %) Let $\mathfrak{a}$ be an ideal of R. Show that

$$M/\mathfrak{a}M \simeq M \otimes_R R/\mathfrak{a}$$

as $R/\mathfrak{a}$-modules.

(ii) (5 %) Let $\mathfrak{p}$ be a prime ideal of R. Show that

$$M_\mathfrak{p} \simeq M \otimes_R R_\mathfrak{p}$$

as $R_\mathfrak{p}$-modules.

(2) An R-module M is said to be torsion free if the annihilator $Ann(x) = 0$ for every non-zero element $x \in M$.

(a) (7 %) Prove the equivalence of the following statements.

(i) M is torsion free, (ii) $M_\mathfrak{p}$ is torsion free for every prime ideal $\mathfrak{p}$ of R, (iii) $M_\mathfrak{m}$ is torsion free for every maximal ideal $\mathfrak{m}$ of R.

(b) (7 %) Let N be a free R-module. Show that every short exact sequence

$$0 \to N' \to M \to N \to 0$$

of R-modules splits and $M \simeq N' \oplus N$ as R-modules. Does it true that the above short exact sequence still split if N is not free?

(3) Let k be a field and let $n \geq 1$ be a positive integer. Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n-indeterminates.

(a) (8 %) Let $f \in A$ be an irreducible polynomial. Show that $A_{(f)}$ is a discrete valuation ring, where $A_{(f)}$ denotes the localization of A at the ideal (f) generated by f.

Note. A ring R is called a *discrete valuation ring* if there exists a discrete valuation $v : K^* \to \mathbb{Z}$ from the field of fraction K of R satisfying (i) $v(xy) = v(x) + v(y)$ and (ii) $v(x+y) \geq \min(v(x), v(y))$ for all $x, y \in K^* = K \setminus \{0\}$ such that R is the valuation ring of v (meaning $R = \{x \in K^* \mid v(x) \geq 0\} \cup \{0\}$).

(b) Let $g_i(x_i) \in \mathbb{Q}[x_i]$, $i = 1, \ldots, n$ ($x_1, \ldots, x_n$ are indeterminates) be irreducible polynomials.

(i) (7 %) Is it true that the ideal $(g_1, g_2, \ldots, g_i)$ prime in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ for all $1 \leq i \leq n$? Give a proof if your answer is yes; otherwise, give a counter example.

(ii) (8 %) Let L_i be a splitting field of $g_i(x_i)$. Assume that $g_i(x_i)$ is irreducible over the field $L_1 L_2 \cdots L_{i-1}$ for every $i > 1$. Is it true that the ideal $(g_1, g_2, \ldots, g_i)$ prime in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ for all $1 \leq i \leq n$? Give a proof if your answer is yes; otherwise, give a