4. Let K be a field and let $\overline{K}$ be a fixed algebraic closure of K. In the following questions, F_1, F_2, K_1 and K_2 are fields in $\overline{K}$. Suppose that F_1 is a Galois extension of K_1 and F_2 is a Galois extension of K_2. Prove or give a counter example of the following:

   (a) (7 pts.) $F_1 \cap F_2$ is a Galois extension of $K_1 \cap K_2$.

   (b) (8 pts.) $F_1 F_2$ is a Galois extension of $K_1 K_2$.

5. Let R be the polynomial ring $\mathbb{Z}[x]$. Fix a prime number p in $\mathbb{Z}$. Suppose that $f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i \in \mathbb{Z}[x]$, where $p \mid a_i$ for all $0 \leq i \leq n-1$. Denote the ring $\mathbb{Z}[f(x)]$ by D.

   (a) (5 pts.) Prove that R is integral over D and prove that D is integrally closed.

   (b) (5 pts.) Prove that R is a free D-module with basis $\{1, x, \ldots, x^{n-1}\}$.

   (c) (5 pts.) Prove that the ideal $(p, f(x))$ generated by p and $f(x)$ in D is a maximal ideal.

   (d) (5 pts.) Consider the polynomial $F(T) = f(T) - f(x)$ in $D[T]$. Prove that $F(T)$ is irreducible in $D[T]$.

   (e) (7 pts.) Let K be the quotient field of D and L be the splitting field of $F(T)$ over K. Prove that $x \in L$ and $[L : K] > n$.

   (f) (8 pts.) Let Γ be the Galois group of L over K. Suppose that $g(x) \in R$ satisfies $g(x^\tau) = g(x)$, $\forall \tau \in \Gamma$. Prove that there exists a polynomial $h(x) \in \mathbb{Z}[x]$ such that $g(x) = h(f(x)) \in D$.