Algebra

Qualify Examination, Summer 2010

(20%) 1. Let G be a finite group with order $o(G) = p^4 q$, where p, q are distinct primes. Suppose that its center $Z(G)$ has order q.

(a) Prove or disprove that any p-Sylow subgroup of G is isomorphic to $G/Z(G)$.

(b) Prove or disprove that the group G is solvable.

(15%) 2. Let $A = \mathbb{Z}[\sqrt{-1}]$. Prove or disprove that the polynomial ring $A[x]$ is Noetherian.

(15%) 3. A ring is said to be simple if it has no ideal other than $\{ 0 \}$ or itself. Let R be the ring of all $n \times n$ matrices over a field. Prove or disprove that R is a simple ring.

(15%) 4. Let G be the Galois group of the polynomial $f(x) = x^4 - 2$ over $\mathbb{F}_5$, where $\mathbb{F}_5$ is the finite field with 5 elements. Is G abelian? Prove your answer.

(15%) 5. Let R be an integral domain containing a field k as a subring. Suppose that R is a finite dimensional vector space over k under the ring multiplication. Prove or disprove that R is a field.

(20%) 6. Let A be a commutative ring and let M be an A-module. For a prime ideal $\wp$ of A, denote $M_\wp = S^{-1}M$, where $S = A \setminus \wp$.

(a) Show that the natural map $M \to \prod M_\wp$ is injective. Where the product ranges over all maximal ideals of A.

(b) Show that a sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact if and only if the sequence $0 \longrightarrow M'_\wp \longrightarrow M_\wp \longrightarrow M''_\wp \longrightarrow 0$ is exact for all prime ideals $\wp$.