國立中央大學數學系
博士班資格考試《分析》試題
2001 年 9 月

15% 1. If $(X, \mathcal{M}, \mu)$ is a measure space and if $f \in L^r(X, \mathcal{M}, \mu)$ for some $0 < r < \infty$. Show that $\lim_{p\to 0}\int_X |f|^p \, d\mu = \mu(\{x \in X \mid f(x) \neq 0\})$.

15% 2. Use the function $y \sin x \, e^{-xy}$ to show the following:

$$\int_0^\infty \frac{\sin x}{x}\left(\frac{1-e^{-x}}{x} - e^{-x}\right)dx = \frac{1}{2}\log 2.$$

15% 3. Assume $f \in L^r$ for some $r < \infty$. Show that $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$.

15% 4. Let $\mathcal{B}_n$ be the smallest σ-algebra containing all open sets in $\mathbf{R}^n$ and let $\mathcal{M}_n = \{E \in \mathbf{R}^n \mid \exists A, B \in \mathcal{B}_n \text{ s.t. } A \subset E \subset B \text{ and } m_n(B-A) = 0\}$, where m_n is the product of n Lebesgue measures on $\mathbf{R}^1$, i. e., $m_n = m_1 \times \ldots \times m_1$. Prove that $\mathcal{B}_i \times \mathcal{B}_j \subset \mathcal{M}_i \times \mathcal{M}_j$ but $\mathcal{B}_i \times \mathcal{B}_j \neq \mathcal{M}_i \times \mathcal{M}_j$.

10% 5. If $\mu, \nu : \mathcal{B}_1 \to [0,\infty)$ are Borel measures with $\mu \times \nu << m_2$. Show that $\dfrac{d(\mu \times \nu)}{d m_2}$ has the form $f(x)g(y)$.

10% 6. Let $I = [a,b]$, $f \in C(I)$, $g \in BV(I)$ and $F(x) = \int_a^x f \, dg$, $a \le x \le b$. Show

(1) F need not be continuous

(2) If g is continuous at x_0 then F is also continuous at x_0.

10% 7. Let $f \in C[0,1]$ and let $f_n(x) = f(x^n)$, $n = 1,2,\ldots$. Assume $\{f_n\}$ is equi-continuous on [0, 1], show that $f \equiv$ constant.

10% 8. Consider the Lebesgue measure space $(\mathbf{R}, \mathcal{M}, m)$. Let $E \in \mathcal{M}$, $m(E) > 0$ and $0 < \alpha < m(E)$. Show that $\exists F \subset E$, $F \in \mathcal{M}$ so that $m(F) = \alpha$.