分析 資格考 2004-FALL

15% 1. Let $1\le p<r<q<\infty$ and suppose that $f\in L^p\cap L^q$. Show that

$$\log\|f\|_r \le \frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}\log\|f\|_p + \frac{\frac{1}{p}-\frac{1}{r}}{\frac{1}{p}-\frac{1}{q}}\log\|f\|_q.$$

15% 2. Assume $f\in L^p(\mathbf{R}^n)$, $1\le p<\infty$. Show that

$$\lim_{y\to 0}\int |f(x+y)-f(x)|^p\,dx = 0.$$

15% 3. Let $f\in L^1(\mathbf{R}^n)$. Prove that $m(\{x|Mf(x)\ge t\})\le \frac{2^n\|f\|_1}{t}$, $0<t<\infty$, where m is the Lebesgue measure and Mf is the Hardy–Littlewood maximal function for f (i.e., $Mf(x)=\sup_{x\in Q}\frac{1}{m(Q)}\int_Q |f(t)|\,dt$).

10% 4. If $\{a_j\}_{j=1}^{\infty}\subset\mathbf{R}$ is a sequence, Show that $\int_0^\infty \sum_{j\le 1/h}|j a_j|\,dh=\sum_{j=1}^{\infty}|a_j|$.

15% 5. Let $I=[0,1]$, $f\in AC(I)$ (i.e., f is absolutely continuous on I), $E\subset I$ and $m(E)>0$. Show that

$$\int_0^1 |f(x)|\,dx \le \int_0^1 |f'(x)|\,dx + \frac{1}{m(E)}\int_E |f(x)|\,dx$$

where m is the Lebesgue measure on I.

15% 6. Let $f\in L^p(\mathbf{R}^n)$, $1\le p<\infty$, $0\le\alpha<\frac{n}{p}$ and let

$$M_\alpha f(x)=\sup_{x\in Q}\frac{1}{\left(m(Q)\right)^{1-\frac{\alpha}{n}}}\int_Q |f(t)|\,dt.$$

Show that $M_\alpha f(x)\le \|f\|_p^{\frac{p\alpha}{n}}\left[M(|f|^p)(x)\right]^{\frac{1}{p}-\frac{\alpha}{n}}$, where Mf is the same function as that defined in problem 3.

15% 7. Consider the Lebesgue measure space $([1,\infty),M,m)$. Define $\mu: M\to[0,\infty]$ by $\mu(E)=\int_E \frac{dx}{x}$. Show that

(1) $\mu << m$, $m << \mu$

(2) $L^\infty(\mu)=L^\infty(m)$

(3) $L^p(m)\subsetneq L^p(\mu)$, $1\le p<\infty$.