(a) Show that $f(x,y)$ is a Lebesgue integrable function of x, whenever $y \in \mathbb{R}$ is fixed and that $\int_{\mathbb{R}} f(x,y)\, d\mathscr{L}^1 x$ is a Lebesgue integrable function of y. (Comment: By the fact that $f(x,y) = f(y,x)$, the same results hold if we switch the order of integrations).

(b) Compute $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\, d\mathscr{L}^1 x \right) d\mathscr{L}^1 y$. (Comment: So $= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\, d\mathscr{L}^1 y \right) d\mathscr{L}^1 x$.)

(c) Comment on the relation between this and Fubini's Theorem.

5. 18 points Let $1 \le p < \infty$ and $E \subset \mathbb{R}^n$. Suppose that $f_k, f \in L^p(E, \mathscr{L}^n)$ and that

$$f_k \longrightarrow f \quad \text{pointwise } \mathscr{L}^n \text{ almost everywhere on } E \text{ as } k \to \infty.$$

(a) Show that $\|f\|_p \le \liminf_{k\to\infty} \|f_k\|_p$.

(b) In case $\|f\|_p = \lim_{k\to\infty} \|f_k\|_p$, prove or disprove that $\lim_{k\to\infty} \|f_k - f\|_p = 0$.

(c) Is possible that $\|f\|_p < \liminf_{k\to\infty} \|f_k\|_p$? Justify your answer!

6. 14 points Let $\varphi \in L^1(\mathbb{R}^n, \mathscr{L}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\, d\mathscr{L}^n x = \alpha \in \mathbb{R}$ and let $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\frac{x}{\epsilon})$, $\epsilon > 0$. If $f \in L^p(\mathbb{R}^n, \mathscr{L}^n)$, then show that the function

$$f_\epsilon(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_\epsilon(y)\, d\mathscr{L}^n y$$

converges to $\alpha f(x)$ in L^p norm.

7. 14 points Let M be the subset of all $f \in L^1([0,1], \mathscr{L}^1)$ such that $\int_{[0,1]} f(t)\, d\mathscr{L}^1 t = 1$. Prove that M is a *closed convex* subset of $L^1([0,1], \mathscr{L}^1)$ which contains *infinitely many* elements of *minimal* L^1 *norm*.