中央大學數學系博士生資格考：分析 (Sep. 01, 2009)

1. Prove or disprove the following statements.
   (a) (10%) For any open subset of $\mathbb{R}$, its Lebesgue measure $|G|$ equals $|\overline{G}|$, where $\overline{G}$ is the closure of G.
   (b) (10%) Let f and g are absolutely continuous functions. Then the composition $f \circ g$ is also an absolutely continuous function.
   (c) (10%) Let f and $\{f_n\}$ be measurable functions which are defined and finite a.e. in a set E with $|E| < \infty$. If $\{f_n\}$ converges in measure on E to f, then $f_n \to f$ a.e. on E.

2. (10%) Construct a subset of $[0,1]$ such that the set is closed, has measure $1/2$, and contains no intervals.

3. (15%) Discuss the following functions of bounded variation:

$$f(x) = \begin{cases} x\sin(1/x), & \text{for } 0 < x \le 1 \\ 0, & \text{for } x = 0. \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2\cos(1/x), & \text{for } 0 < x \le 1 \\ 0, & \text{for } x = 0. \end{cases}$$

4. (10%) Let f be nonnegative measurable function on $\mathbb{R}$ and $p > 0$. Show that

$$\int_{\mathbb{R}} [f(x)]^p dx = p\int_0^\infty t^{p-1}|\{x \in \mathbb{R} : f(x) > t\}|dt.$$

5. Find the Lebesgue integral and limits:
   (a) (7%) $\displaystyle\int_0^1 \frac{\ln(1-x)}{x}dx$;
   (b) (8%) $\displaystyle\lim_{n\to\infty} (L)\int_0^\infty \frac{1}{(1+x/n)^n x^{1/n}}dx.$

6. (10%) Let $f \in L^p(\mathbb{R}^n), 1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Prove that

$$\|f\|_p = \sup_{\|g\|_{p'} \le 1} \int_{\mathbb{R}^n} f(x)g(x)\,dx.$$

7. (10%) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces. Suppose h and g are integrable functions on X and Y, and define $f(x,y) = h(x)g(y)$. Show that f is integrable on $X \times Y$ and

$$\int_{X\times Y} f\,d(\mu \times \nu) = \int_X h\,d\mu \int_Y g\,d\nu.$$