Department of Mathematics
National Central University

Ph. D. Qualifying Examination

Spring, 2001.

Analysis

Answer all of the following questions.

1. Suppose σ is a one-to-one transformation of the set of positive integers onto itself and a_n is a non-negative real number for all $n = 1, 2, \ldots$. Prove that
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}.$$

2. Let P be the class of all polynomials of the form
$$p(x) = \sum_{j=0}^{n} a_j x^{2j+1}.$$
Show that a continuous function f in $C[0,1]$ can be uniformly approximated on $[0,1]$ by a sequence in P if and only if $f(0) = 0$.

3. Suppose $f_n \in L^p(\mathbb{R})$, $||f_n||_p \leq M < \infty$, $n = 1, 2, \ldots$, where $1 < p < \infty$. Suppose f_n converges to f almost everywhere and that $g \in L^q(\mathbb{R})$, where $1/p + 1/q = 1$. Show that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)dx = \int_{\mathbb{R}} f(x)g(x)dx.$$

4. Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous and of bounded variation. Assume that for each $\epsilon > 0$, $f_{|[\epsilon,1]}$ is absolutely continuous. Show that f is absolutely continuous on $[0,1]$.