Ph. D. Qualifying Examination, 2003

Analysis

Choose any 5 problems in the following test.

√ 1.(a) Prove that every function of bounded variation has at most a countable number of discontinuities.

(b) Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0, 1]$, but the variation of $f, V[f; 0, 1] = +\infty$.

√ 2. For $1 \leq p < \infty$, we define the spaces $(l^p, ||\cdot||_p)$ and $(l^\infty, ||\cdot||_\infty)$ as the following respectively

$$l^p = \{<x_i>_{i=1}^{\infty} : (||<x_i>||_p)^p = \sum_{i=1}^{\infty} |x_i|^p < \infty, x_i \in R \ \forall i\},$$
$$l^\infty = \{<x_i>_{i=1}^{\infty} : ||<x_i>||_\infty = \sup |x_i| < \infty, x_i \in R \ \forall i\}.$$

Prove that l^p is complete and l^∞ is a Banach space.

3. Let v_n be the volume of the unit ball in R^n. Show by using Fubini's theorem that

$$v_n = 2v_{n-1} \int_0^1 (1-t^2)^{\frac{n-1}{2}} \, dt.$$

√ 4. Prove Egoroff's Theorem : If $< f_n >$ is a sequence of measurable functions that converge to a real-values function f a.e. on a measurable set E of finite measure, then given $\eta > 0$, there is a subset $A \subset E$ with $m(A) < \eta$ such that f_n converges to f uniformly on $E \backslash A$.

√ 5. Let $1 \leq r,\ p_1, p_2, \cdots, p_k \leq \infty$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{r}$. Prove that : if $f_1 \in L^{p_1}, \cdots, f_k \in L^{p_k}$, then

$$||f_1 \cdots f_k||_r \leq ||f_1||_{p_1} \cdot ||f_2||_{p_2} \cdots ||f_k||_{p_k}.$$