中央大學數學系博士生資格考：分析 (Aug. 27, 2010)

1. (15%) Let f be a real-valued function defined for all real numbers. Prove that the set of points at which f is continuous is a G_δ.

2. (15%) If $\{f_k(x)\}_{k=1}^{\infty}$ is a sequence of measurable functions. Show that the set of points at which $\{f_k(x)\}_{k=1}^{\infty}$ diverges is measurable.

3. Prove or disprove the following statements.

   (a) (10%) Let f be of bounded variation and continuous on $[a,b]$, then f is absolutely continuous on $[a,b]$.

   (b) (10%) The function $f(x) = \begin{cases} x^2\sin(1/x^2), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$ is absolutely continuous on $[-1,1]$.

   (c) (10%) Let f and $\{f_n\}$ be measurable functions which are defined and finite a.e. in a set E with $|E| < \infty$. If $f_n \to f$ a.e. on E, then $\{f_n\}$ converges in measure on E to f.

   (d) (10%) Let $f, \{f_n\} \in L^p$. If $f_n \to f$ a.e. and $\|f_k\|_p \to \|f\|_p, 1 \le p < \infty$, then $\{f_n\}$ converges to f in L^p.

4. (15%) Let $0 < p_1, p_2 < \infty$ and $f \in L^{p_1} \cap L^{p_2}$. Show that $f \in L^p, p_1 \le p \le p_2$.

5. (15%) Let $f \in L(\mathbb{R}^n)$ and $g \in L(\mathbb{R}^n)$. Prove that $f*g \in L(\mathbb{R}^n)$ and $\|f*g\|_1 \le \|f\|_1\|g\|_1$. Here $(f*g)(x)$ is defined by

$$(f*g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$