Ph.D. Qualifying Examination: Analysis (2010.01)

Stage Setting: In the following problems, whenever not specified, the functions are assumed be real-valued.

1. (10%) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. If $f = 0$ almost everywhere with respect to Lebesgue measure, prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

2. (10%) Let $f : [0,1] \to \mathbb{R}$ be defined by setting $f(0) = 0$ and $f(x) = x^2 \cos(1/x^2)$ if $x \in (0,1]$. Determine whether f is absolutely continuous on $[0,1]$. Prove your answer.

3. (15%) Let $(X, \mathcal{B}, \mu)$ be a finite measure space. For every pair of measurable functions f and g let
$$d(f,g) = \int_X \frac{|f-g|}{1+|f-g|} d\mu.$$
   (a) Show that a sequence $\{f_n\}$ of measurable functions satisfies $f_n \to f$ in measure if and only if $\lim_{n\to\infty} d(f_n, f) = 0$.

   (b) Denote by Y the space of all equivalence classes $\overline{f}$ of measurable functions on $(X, \mathcal{B}, \mu)$, with $\overline{f} = \overline{g}$ if and only if $f = g$ almost everywhere. Prove that (Y, d) is a complete metric space.

4. (10%) For each n let the function $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx^{n-1}}{1+x}$ for all $x \in [0,1]$. Then show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2}$.

5. (15%) Let $(X, \mathcal{B}, \mu)$ be a finite measure space and let $f : X \to \mathbb{R}$ be a measurable function. If f^n is integrable for each n, then show that $\int_X f^n d\mu = c$ (= a constant) holds for all $n = 1, 2, \ldots$ if and only if $f = \chi_A$ for some measurable subset A of X.

6. (10%) If two functions $f, g \in L^3(X, \mathcal{B}, \mu)$ satisfy
$$\|f\|_3 = \|g\|_3 = \int_X f^2 g d\mu = 1,$$
then show that $g = |f|$ almost everywhere holds.

7. (15%) Let $(X, \mathcal{B})$ and (Y, Σ) be two measurable spaces and μ a measure on $\mathcal{B}$. Assume that $T : X \to Y$ has the property that $T^{-1}(A) \in \mathcal{B}$ for each $A \in \Sigma$. Let ν be a measure on Σ defined by $\nu(A) = \mu(T^{-1}(A))$ for each $A \in \Sigma$.

   (a) If $f \in L^1(\nu)$, then show that $f \circ T \in L^1(\mu)$ and $\int_Y f d\nu = \int_X f \circ T d\mu$.

   (b) If μ is finite and ω is a σ-finite measure on Σ such that $\nu \ll \omega$, then show that there exists a function $g \in L^1(\omega)$ such that $\int_X f \circ T d\mu = \int_Y f g d\omega$ holds for each $f \in L^1(\nu)$.

8. (15%) Let $\ell^2 = \{(a_1, a_2, \ldots) : \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$ be a normed space with the norm $\|(a_1, a_2, \ldots)\|_2 = (\sum_{n=1}^{\infty} |a_n|^2)^{1/2}$.

   (a) Let $U = \{(a_1, a_2, \ldots) \in \ell^2 : \|(a_1, a_2, \ldots)\|_2 \leq 1\}$. Determine whether U is compact in $(\ell^2, \|\cdot\|_2)$, prove your answer.

   (b) Let $A = \{(a_1, a_2, \ldots) \in \ell^2 : |a_n| \leq \frac{1}{n} \text{ for all } n\}$. Determine whether A is compact in $(\ell^2, \|\cdot\|_2)$, prove your answer.