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NATIONAL CENTRAL UNIVERSITY

Real Analysis Ph.D. Qualifying Exam

August 27th, 2007

There are 7 question sets of total 100 points.

1. [12 %] Let $A \subset \mathbb{R}^n$ with positive Lebesgue measure $\mathcal{L}^n(A)$. Prove or disprove that for any $0 < \theta < \mathcal{L}^n(A)$, there is a compact $K \subset A$ such that $\mathcal{L}^n(K) = \theta$.

2. [12 %] Let $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a sequence of Lebesgue measurable functions. Prove that the set of points in $\mathbb{R}^n$ for which $\{f_k\}$ converges is a Lebesgue measurable subset of $\mathbb{R}^n$.

3. [16 %] Let $I = [0,1] \times [0,1]$ and $f(x,y) = \begin{cases} (xy-1)^{-1}, & \text{if } xy - 1 \neq 0; \\ 0, & \text{if } xy - 1 = 0. \end{cases}$ Find all $p > 0$ so that $f \in L^p(I, \mathcal{L}^2.)$.

4. [12 %] Compute, for $t > 0$, $\lim_{n\to\infty} \int_{[0,n]} \left(1 - \frac{x}{n}\right)^n x^{t-1}\, d\mathcal{L}^1(x) =$? Give reasons for the steps you take in your computation.

5. [16 %] Suppose that $f_k \to f$ a.e. and that $f_k, f \in L^p(\mathbb{R}^n, \mathcal{L}^n), 1 < p < \infty$. If the L^p norm $\|f_k\|_p \leq M < +\infty \quad \forall k \in \mathbb{N}$, show that $\int f_k g\, d\mathcal{L}^n \to \int fg\, d\mathcal{L}^n, \quad \forall g \in L^{p'}$. Here $p' = \frac{p-1}{p}$ is the conjugate exponent of p. What happen if $p = 1$?

6. [18 %] Let $p \in (1, \infty)$. Prove that the unit ball of $L^p(\mathbb{R}^n, \mathcal{L}^n)$, which is the set $\{f \in L^p(\mathbb{R}^n, \mathcal{L}^n) : \|f\|_p \leq 1\}$, is strictly convex: *i.e.* If $f, g \in L^p(\mathbb{R}^n, \mathcal{L}^n)$, $\|f\|_p = \|g\|_p = 1$, $f \neq g$, and $h = \frac{f+g}{2}$, then $\|h\|_p < 1$. Show that the result fails to hold when $p = 1$ or ∞.

7. [14 %] Let $f : \mathbb{R}^n \longrightarrow [0, \infty)$ be Lebesgue integrable and $f \equiv 0$ outside a ball of positive radius on $\mathbb{R}^n$. Show that there is $b \in \mathbb{R}^n$ so that

$$b \cdot v \int_{\mathbb{R}^n} f(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^n} (x \cdot v) f(x)\, d\mathcal{L}^n(x) \qquad \text{for } v \in \mathbb{R}^n.$$