Department of Mathematics
National Central University

Ph. D. Qualifying Examination

Fall, 2000.

Real Analysis

1. Let f be a continuous function one-to-one from a compact space X onto a Hausdorff space Y. Prove that f is an homeomorphism.

2. Let X and Y be metric spaces. Let $\{f_n\}_n$ be a sequence of continuous functions from X into Y which converge to a function f uniformly on each compact subset of X. Show that f is continuous.

3. Let X and Y be compact Hausdorff spaces. Show that for each continuous real-valued function f on $X \times Y$ and each $\epsilon > 0$, there are continuous real-valued functions $g_1, \ldots, g_n$ on X and $h_1, \ldots, h_n$ on Y such that for each $(x, y) \in X \times Y$ we have

$$\left| f(x,y) - \sum_{i=1}^{n} g_i(x) h_i(y) \right| < \epsilon.$$

4. Prove that if a real-valued function f is integrable on $[a, b]$ and

$$\int_a^x f(t)\, dt = 0$$

for all x in $[a, b]$ then $f(t) = 0$ a.e. in $[a, b]$.

5. (a) Show that the set $\mathbb{R}^{n-1} \times \{0\}$ has measure zero in $\mathbb{R}^n$.

   (b) Let $[0,1]^2 = [0,1] \times [0,1]$. Let $f : [0,1]^2 \longrightarrow \mathbb{R}$ be defined by setting $f(x,y) = 0$ if $y \neq x$ and $f(x,y) = 1$ if $y = x$. Show that f is integrable over $[0,1]^2$.