NCU Ph.D Qualification Exam. Differential Geometry

January 19 2010

State your assumptions or cite a known theorem if you are using it. Present your solutions as simple as possible yet rigorous enough.

I) **Definitions and Concepts** (10 ps). Give the following definitions, there may be more than one definitions involved in some questions. In this case, give all relevant definitions.

(i) A differentiable manifold.

(ii) Tangent vectors and the differential dF_p of a differentiable function $F : M \to N$ where M and N are differentiable manifolds and $p \in M$.

(iii) A Riemannian/Levi-Civita connection. (There may be more than one definitions involved here i.e. what is a connection etc... what does it mean for a connection to be...).

(iv) An isometry between two Riemannian manifolds.

(v) The exponential map $exp : TM \to M$.

II) **Examples** (15 ps).

(i) Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. describe three different local coordinate chart system on $\mathbb{S}^2$.

(ii) Give an example of a (abstract) manifold in the sense that it is in the form of a quotient space and describe its local coordinates neighborhoods.

(iii) Describe the hyperbolic space (hyperbolic metric) of dimension two.

III) **Theorems and Proofs**. In what follows, just do the best that you can (even if you cannot finish the problem completely).

A) (20 ps) State and prove the Gauss lemma.

B) (20 ps) State and prove the Fundamental Theorem of Riemannian Geometry (or Levi-Civita's theorem).

C) (20 ps) Let $N : S \to \mathbb{S}^2$ be the Gauss map of an orientable regular surface $S \subset \mathbb{R}^3$ with an orientation. Show that the differential of the Gauss map $dN_p : T_p(S) \to T_p(S) \equiv T_{N(p)}\mathbb{S}^2$ is a self-adjoint linear map. Note that we have identified the tangent plane $T_{N(p)}\mathbb{S}^2$ with $T_p(S)$ since they are parallel planes.

D) (15 ps) Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parameterized by arc-length. Suppose that $\tau(s) \neq 0$, where $\tau(s)$ is the torsion of α and $k'(s) \neq 0$ where $k(s) = |\alpha''(s)|$ is the curvature of α. Prove that $\alpha(I)$ lies on a sphere if and only if

$$R^2 + (R')^2T^2 \equiv const\ ,$$

where $R = 1/k$, $T = 1/\tau$ and R' is the derivative of R with respect to s. Recall that given $\alpha(s)$ a regular curve (that is $\alpha'(s) \neq 0$), we have $t(s) = \alpha'(s)$ is the unit tangent of α at s, $t'(s) = k(s)n(s)$, here n is the normal and $b(s) = t(s) \wedge n(s)$ is the binormal. If $\alpha''(s) \neq 0$, the number $\tau(s)$ for which $b'(s) = \tau(s)n(s)$ is called the torsion of α at s.