敬請公布

Differential Equations (任選5題作答)

1. (20%) Consider the following initial value problem (IVP)

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)), \quad f : D \subseteq R \times R^n \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where D is an open set of $R \times R^n$ containing (t_0, x_0). Prove the following statements respectively :

(a) Let $f \in C(D), (t_0, x_0) \in D$. Then the (IVP) has a solution on an interval $I = [t_0 - c, t_0 + c]$.

(b) If, in addition, $f(t, x)$ satisfies Lipschitz condition in x, then the solution of (IVP) is unique.

2. (20%) Find the general solution of $x' = Ax$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{pmatrix}$$

3. (20%) Consider the following Predator-Prey system

$$\begin{cases} x' = \gamma x(1 - \dfrac{x}{K}) - \alpha xy \\ y' = y(\beta x - d), \ \gamma, K, \alpha, \beta > 0 \\ x(0) > 0, y(0) > 0. \end{cases} \tag{2}$$

(a) Show that the solutions $x(t), y(t)$ of Eq. (2) are defined for all $t > 0$ and the solutions are positive and bounded for all $t > 0$.

(b) Do the stability analysis for each equilibrium.

4.(a) Consider the system

(20%)

$$\begin{cases} \dfrac{dx}{dt} = f(t, x), \\ \dfrac{dy}{dt} = g(t, x). \end{cases} \tag{3}$$

If $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is of one sign in a simple-connected domain D in R^2. Prove that Eq. (3) does not have any periodic solution in D.

(b) Show that $x'' + f(x)x' + g(x) = 0$ cannot have any periodic solution whose path in a region where $f(x)$ is on one sign.

5. (20%) Let $p(t + \pi) = p(t) \not\equiv 0$ for all t and $0 \leq \int_0^\pi p(t)dt \leq \frac{4}{\pi}, p(t)$ is xontinuous on R. Prove all solutions of the equation $u'' + p(t)u = 0$ are bounded.

6. (20%) Consider the van der Pol equation

$$x'' + \epsilon(x^2 - 1)x' + x = 0. \tag{4}$$

Prove the following statements.

(a) Equation (4) has a unique asymptotically stable limit cycle Γ for every $\epsilon > 0$. (State the theorem used.)

(b) Let $D = \{(x, y) | x^2 + y^2 < 3\}$. Then $\Gamma \not\subset D$. (State the theorem used.)