國立中央大學數學系
博士班資格考試
〈圖論〉試題
Fall 2001

注意: 請寫下解題的詳細過程，並請證明你在解題過程中所用到的所有定理。

Usage of Greek alphabet

$\alpha(G)$: independence number. $\chi(G)$: chromatic number. $\kappa(G)$: vertex connectivity. $\kappa'(G)$: edge-connectivity. $\delta(G)$: minimum degree. $e(G)$: number of edges.

Problem 1 Show that if x is a vertex of an n-vertex tree G, then

$$\sum_{y \in V(G)} d(x,y) \leq \binom{n}{2}$$

. (10 分)

Problem 2 Prove that the center of a tree T is one vertex if and only if

$$\mathrm{diam}(T) = 2\ \mathrm{rad}(T).$$

(10 分)

Problem 3 Prove that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. And show that there exists a graph H with $\kappa(H) < \kappa'(H) < \delta(H)$. (10 分)

Problem 4 (20 分)

(a) Prove or disprove that if G is a simple graph with number of vertices $n(G) \geq 3$, and G has at least $\alpha(G)$ vertices of degree $n(G) - 1$, then G is Hamiltonian.

(b) Prove that if $\kappa(G) \geq \alpha(G)$, then G has a Hamiltonian cycle (unless $G = K_2$).

Problem 5 (20 分)

(a) Prove that a graph G has a 1-factor if and only if $o(G - S) \leq |S|$ for every $S \subseteq V(G)$.

(b) Using (a) to show that every 3-regular graph with no cut-edges has a 1-factor.

Problem 6 Prove that $\chi(G) \cdot \chi(\bar{G}) \geq n(G)$, use this to prove that $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{n(G)}$. (10分)

Problem 7 Prove by induction on $e(G)$ that a plane graph is bipartite if and only if every face has even length. (10 分)

Problem 8 Show that the Ramsey numbers $R(3,3) = 6$ and $R(3,4) = 9$. (10 分)