注意: 請寫下解題的詳細過程,並盡可能證明所有使用到的定理與引理.

Usage of Greek alphabet

$\delta(G)$: minimum degree.

$\kappa(G)$: vertex connectivity.

$\kappa'(G)$: edge-connectivity.

$\chi(G)$: chromatic number.

Problem 1 Show that every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$ (provided that $\delta(G) \geq 2$). (20 分)

Problem 2 Show that if T is a tree with m edges and G is a simple graph with $\delta(G) \geq m$, then T is a subgraph of G. (20 分)

Problem 3 Prove that every k-regular bipartite graph has a perfect matching. (20 分)

Problem 4 Prove that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. And show that there exists a graph H with $\kappa(H) < \kappa'(H) < \delta(H)$. (20 分)

Problem 5 Show that if G is a planar graph, then $\delta(G) \leq 5$ and $\chi(G) \leq 6$. (20分)