Graph Theory

August 27, 2010

1. (10%) Let $\tau(G)$ denote the number of spanning trees of a graph G. Find the value of $\tau(K_n - e)$ and prove your result by using Cayley's Formula, $\tau(K_n) = n^{n-2}$.

2. If C is a cycle, and e is an edge connecting two nonadjacent vertices of C, then we call e a *chord* of C. Prove or disprove the following statements.

   (a) (5%) If every vertex of a graph G has degree at least 3, then G contains a cycle with a chord.

   (b) (10%) If G has $n \geq 4$ vertices and $2n - 3$ edges, then G contains a cycle with a chord.

3. G is a graph with n vertices where $n \geq 4$.

   (a) (15%) Suppose $\deg x + \deg y \geq n$ for every nonadjacent vertices x, y of G. Show that G contains a Hamiltonian cycle.

   (b) (5%) Suppose $\deg x + \deg y \geq n - 1$ for every nonadjacent vertices x, y of G. Show that G contains a Hamiltonian path.

   (c) (5%) Suppose G contains at least $\binom{n-1}{2} + 2$ edges. Show that G contains a Hamiltonian cycle.

4. (20%) Let $\alpha'(G)$ denote the maximum size of matching in a graph G and $\beta'(G)$ be the minimum size of edge cover of G. Prove that if G has n vertices without isolated vertices, then $\alpha'(G) + \beta'(G) = n$.

5. (10%) $\chi(G)$ is the least k such that G is k-colorable and the maximum degree is denoted by $\Delta(G)$. Prove that $\chi(G) \leq \Delta(G) + 1$.

6. (10%) G is a planar graph. Show that G has a vertex of degree at most 5.

7. (10%) Show that among six persons it is possible to find three mutual acquaintances or three mutual non-acquaintances.