Numerical Analysis **Qualifying Exam** **27 Aug. 2010**

If you think that a problem has been stated incorrectly, indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial. Each problem is 20 points.

1. Let$(\mathcal{K}, \mathcal{P}, \mathcal{N})$ be a finite element, where $\mathcal{K}$ is a domain with piecewise smooth boundary, $\mathcal{P}$ is a finite-dimensional space of functions on $\mathcal{K}$, and $\mathcal{N} = \{N_1, N_2, \dots N_k\}$ are the nodal variables. Now let $\mathcal{K}$ is unit square in $\mathcal{R}^2$, $\mathcal{P} = \mathcal{Q}_1$ (bilinear functions), and $\mathcal{N}$ are $(1/2, 0), (1, 1/2), (0, 1/2), (1/2, 1)$. Do the degrees of freedom at $\mathcal{N}$ determine $\mathcal{P}$, justify your answers.

2. Consider the boundary value problem

$$-u'' + u' + u = f \quad \text{on } [0,1], \quad u'(0) = u'(1) = 0.$$

Take

$$V = H^1(0,1)$$
$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\,dx$$
$$F(v) = (f, v)$$

Prove that $a(\cdot,\cdot)$ is continuous and V-elliptic (coercive). If the above differential equation is changed to

$$-u'' + ku' + u = f.$$

Show that $a(\cdot,\cdot)$ need not be coercive for large k.

3. Let $a(\cdot,\cdot)$ be the inner product for a Hilbert space V. Prove that the following two statements are equivalent for $F \in V'$ and an arbitrary (closed) subspace U of V:

(a) $u \in U$ satisfies $a(u,v) = F(v) \quad \forall v \in V$

(b) u minimizes $\frac{1}{2}a(v,v) - F(v)$ over $v \in V$,

i.e. show that existence in one implies existence in the other.

4. Show that the three-dimensional ADI method for

$$u_t = A_1 u + A_2 u + A_3 u$$

given by

$$\begin{aligned}
(I - \frac{k}{2}A_{1h})\tilde{v}^{n+1/3} &= (I + \frac{k}{2}A_{3h})v^n \\
(I - \frac{k}{2}A_{2h})\tilde{v}^{n+2/3} &= (I + \frac{k}{2}A_{2h})v^{n+1/3} \\
(I - \frac{k}{2}A_{3h})\tilde{v}^{n+1} &= (I + \frac{k}{2}A_{1h})v^{n+2/3}.
\end{aligned}$$

is equivalent to

$$\begin{aligned}
& (I - \frac{k}{2}A_{1h})(I - \frac{k}{2}A_{2h})(I - \frac{k}{2}A_{3h})v^{n+1} \\
= \quad & (I + \frac{k}{2}A_{1h})(I + \frac{k}{2}A_{2h})(I + \frac{k}{2}A_{3h})v^n.
\end{aligned}$$

only if the operators A_{1h} and A_{2h} commute.

5. Consider the heat equations

$$\begin{aligned}
& u_t = u_{xx},\ t \geq 0,\ 0 \leq x \leq 1, \\
& u(0,t) = u(1,t) = 0,\ t \geq 0, \\
& u(x,0) = f(x),\ x \in [0,1],
\end{aligned}$$

where $f(x)$ is continuous.

(a)Construct an explicit finite element scheme in solving the system numerically.
(b)Discuss the convergent and stability properties of your scheme.
(c)Construct an implicit finite element scheme to improve the stability of (a).

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